Noise-induced transitions in state-dependent dichotomous processes

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In a number of stochastic systems the random forcing is represented as a dichotomous Markov noise. A common characteristic of these models is that the noise is usually supposed to be independent of the state of the forced dynamical system. However, there are several situations in which positive or negative feedback exist between the system and the random driver. This paper investigates a class of systems characterized by feedback between dichotomous Markov noise and the system's dynamics. The effect of the feedback is accounted for through a state dependency in the transition rates of the dichotomous noise. We study noise-induced transitions in these systems, with special attention to the delicate problem of correctly defining the deterministic counterpart of the stochastic system. We find that (i) if in the absence of any feedback the dynamical system has a single deterministic stable point, the deterministic dynamics remain monostable when a negative feedback is introduced, while they may become bistable in the presence of a positive feedback. (ii) Noise may induce bistability in the presence of a null or negative feedback. (iii) Bistable deterministic dynamics, induced by the positive feedback, may be destroyed by the noise, which tends to stabilize the system around a new intermediate stable state between those of the deterministic dynamics.

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I. INTRODUCTION

The dichotomous Markov noise (DMN) is a stochastic process described by a variable, $\xi_{dn}(t)$, that can take only two values, say Δ_1 and Δ_2, with the transition $\Delta_1 \rightarrow \Delta_2$ occurring at rate k_1, and $\Delta_2 \rightarrow \Delta_1$ at rate k_2. In spite of its simple structure, the dichotomous noise has drawn the attention of a number of authors (e.g., see the recent review by Bena [1]) for two main reasons. The first reason is that the dichotomous noise is a simple, analytically tractable form of colored noise; in fact, it is possible to obtain exact analytical solutions for a stochastic differential equation driven by DMN, in particular in steady state conditions [1–4]. Therefore, the first reason why the DMN is used pertains to what we here define as the *functional* usage of the DMN, i.e., to its function as a tool to conveniently represent a correlated random forcing. Thus, in this case (functional usage) the starting point is a given deterministic system, say $dx/dt=f(x)$, and DMN is typically used to investigate the effect of a zero-mean correlated random driver in this system. Beside its analytical tractability, generality is another property justifying the functional usage of DMN: in fact, both Poisson noise and Gaussian noise can be recovered from the dichotomous noise by taking suitable limits [2]. The second reason behind the success of the DMN is that a broad class of systems that randomly switch between two dynamical states can be represented through the use of the DMN. This approach is what we will denote here as the *mechanistic* usage of the DMN, wherein the DMN is used to represent a dynamical behavior, i.e., the mechanisms of random switching between two states.

This distinction in the manner of how the DMN is used may have important consequences, in particular when noise-induced transitions in systems driven by DMN are considered. Noise-induced transitions are associated with the emergence of new ordered states as the noise intensity (i.e., the noise variance) exceeds a critical threshold [3]. The randomness of an external driver is then able to profoundly affect the dynamical properties of the system, by inducing bifurcations that do not exist in the underlying deterministic dynamics. To correctly identify noise-induced transitions one should then be able to define the deterministic counterpart of the dynamics, which indeed depends on the way the DMN is used (functional or mechanistic interpretation). One of the aims of the present paper is to clarify this point, which is not fully manifest in the literature.

The other aim of the paper is to extend some of the existing results on the analysis of dynamical systems forced by dichotomous noise, which exhibit noise-induced transitions [3–5]: in fact, previous studies have concentrated on the study of the interaction of multiplicative noise with nonlinear dynamics, and in particular on the case in which the multiplicative noise can be factorized, i.e., expressed as the byproduct of a noise term, ξ_{dn}, with a function of the state variable. Here we show that noise-induced transitions may also emerge from a state dependency of the noise parameters, i.e., from the dependency of the rates k_1 and k_2 on the state variable. The topic is relevant, not only from the speculative viewpoint. In fact, most natural systems are forced by stochastic environmental fluctuations, which determine the random alternation of conditions favorable for the growth or the decay of indicators of the health of the system (e.g., total biomass, vegetation cover, or biodiversity). In a number of cases the random environmental fluctuations depend on these state variables. In fact, positive or negative feedback [6] are commonly found between ecosystem dynamics and their

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limiting resources (e.g., [7–10]) or disturbance regime (e.g., [11–13]). Although the need for the study of a state-dependent version of the dichotomous noise has already been stressed elsewhere [1,14,15], a full exploration of the properties of this process is still lacking.

In the following section we will formulate a framework for the study of univariate systems forced by state-dependent dichotomous Markov noise under the two possible interpretations of mechanistic and functional DMN. Noise induced transitions for this class of processes are then studied in Sec. III. A simple example of dynamical system driven by dichotomous noise will show in Sec. IV how the feedback strongly influences the stochastic dynamics, suggesting that the state dependency of the dichotomous noise should be in general accounted for.

II. MODELING FRAMEWORK

This section is devoted to defining our modeling framework for investigating the interactions between noise and feedback in dynamical systems. We first clarify in Sec. II A the differences between the mechanistic and functional usage of the DMN in the absence of the feedback, and then introduce the state dependency in Sec. II B.

A. Mechanistic vs functional interpretation of the DMN

We start by considering the mechanistic approach, which stems from a class of processes characterized by the following three components: (i) the dynamical system, whose state is expressed by one state variable $x(t)$ (where t is time); (ii) a random driver q, accounting for the uncertainty affecting the system; (iii) a threshold value s of q, marking the transition between conditions favorable to the growth or decay of x. For example, the variable x could represent the vegetation biomass in semiarid environments [16], or the riparian vegetation along a river transect [17]; correspondingly, q could represent the random rainfall fluctuations that determine the occurrence of water-limited conditions or of flooded or unflooded states, respectively. Thus the stochastic driver determines the random alternation between stressed and unstressed conditions for the ecosystem (e.g., [16,17]).

The two alternating dynamics of x involve growth and decay and can be modeled by two functions, $f_1(x)$ and $f_2(x)$, respectively,

$$\frac{dx}{dt} = \begin{cases} f_1(x) & \text{if } q \geqslant s \quad (1a) \\ f_2(x) & \text{if } q < s \quad (1b) \end{cases}$$

with $f_1(x)>0$ and $f_2(x)<0$. Equations (1) are written assuming that q is a resource, in that values of q exceeding the threshold are associated with unstressed conditions (in the sense that x grows). However, the general results presented in this paper do not change when the random driver is a stressor. In this case the conditions in Eqs. (1) are reversed, i.e., growth or decay occur when q is below or above the threshold, respectively.

The class of processes defined in the previous paragraphs can be conveniently represented through a suitable dichotomic Markov process, which naturally leads to our mechanistic usage of the DMN. The driving process is random and switches between two possible states: "success" (or "no stress"), when q is above the threshold, or "failure" (or "stress"), when q is below the threshold (see Fig. 1, continuous line). This is by definition a dichotomic process. If one further supposes that q, which is discrete in time, changes randomly at each time step (i.e., it is uncorrelated), the driving noise is the outcome of a Bernoulli trial with probability of success $k_2=1-P_Q(s)$, where $P_Q(s)$ is the cumulative probability distribution of q, evaluated in $q=s$. The residence time in the "above threshold" state is then an integer number n_1 with a geometric probability distribution, $p_{N_1}(n_1)=k_2^{n_1-1}(1-k_2)$, $n_1=1,\ldots,\infty$, with average $\langle n_1\rangle=\frac{1}{1-k_2}$. Analogously, the residence time in the "below threshold" state is distributed as $p_{N_2}(n_2)=(1-k_2)^{n_2-1}k_2$, $n_2=1,\ldots,\infty$, with average $\langle n_2\rangle=\frac{1}{k_2}$. The dichotomic Markov noise is obtained as the continuous time approximation of this driving process. In fact, in continuous time the residence time in each state becomes exponentially distributed (the exponential distribution is the continuous counterpart of the geometric distribution), which is a basic property of the DMN (see, for example, [1]).

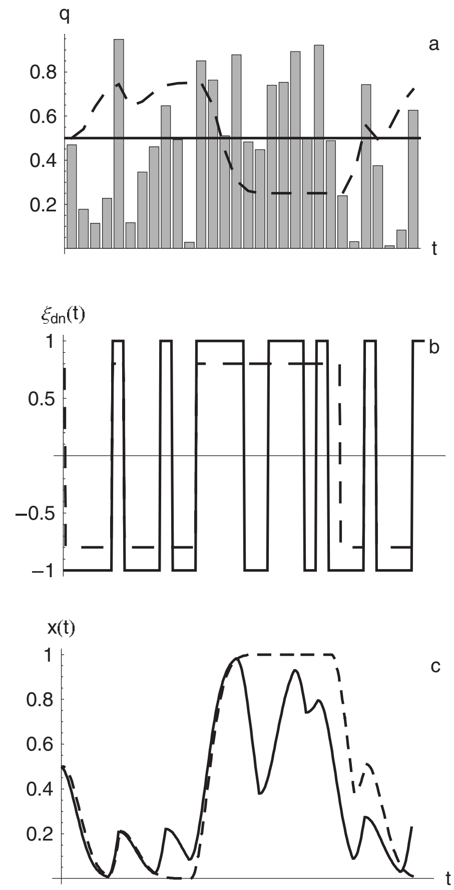

FIG. 1. Example of the relations between the external forcing q [panel (a), gray bars], the threshold s [panel (a), continuous and dashed lines], the corresponding DMN [panel (b)], and the resulting x dynamics [panel (c)]. Continuous and dashed lines correspond to no feedback and positive feedback between x and s, respectively.

The overall dynamics of the variable x can then be expressed by a stochastic differential equation forced by dichotomous Markov noise, $\xi_{dn}(t)$, assuming (constant) values, Δ_1 and Δ_2

$$\frac{dx}{dt} = f(x) + g(x)\xi_{dn}(t) \tag{2}$$

with

$$f(x) = -\frac{\Delta_2 f_1(x) - \Delta_1 f_2(x)}{\Delta_1 - \Delta_2}, \quad g(x) = \frac{f_1(x) - f_2(x)}{\Delta_1 - \Delta_2}. \tag{3}$$

The transition rates are defined by $k_1 = P_Q(s)$ and $k_2 = 1 - k_1 = 1 - P_Q(s)$. As for the values of Δ_1 and Δ_2, in the mechanistic approach the DMN is used as a tool to randomly switch between $f_1(x)$ and $f_2(x)$. The only relevant characteristics of the DMN are in this case the switching rates k_1 and k_2, while all other noise characteristics, included its mean, $\Delta_1 k_2 + \Delta_2 k_1$, and variance, $-\Delta_1\Delta_2$, are not relevant to the representation of the x dynamics. In fact, in this case x switches between two dynamics [modeled by $f_1(x)$ and $f_2(x)$] that are independent of Δ_1 and Δ_2. As a consequence, Δ_1 and Δ_2 may assume arbitrary values.

Under the functional interpretation of the DMN, in contrast, the ingredients of the dynamical system are (i) a dynamical system that deterministically evolves following the differential equation $\frac{dx}{dt} = f(x)$; and (ii) a random colored forcing $\xi(t)$ which acts on the time derivative of x, modulated by a function $g(x)$ of the state variable. The time evolution of the system is therefore expressed by the stochastic differential equation $\frac{dx}{dt} = f(x) + g(x)\xi(t)$. The functional usage of the DMN consists in approximating $\xi(t)$ as a DMN, i.e., $\xi(t) = \xi_{dn}(t)$. In this case neither of the values of k_1, k_2, Δ_1, and Δ_2 is arbitrary and these parameters need to be determined by adapting the DMN to the characteristics of the driving noise (i.e., for example, by matching the mean, variance, skewness, and correlation scale). Moreover, the functions $f(x)$ and $g(x)$ are in this case defined *a priori*, while $f_1(x)$ and $f_2(x)$ change with changing noise characteristics since they are obtained by inverting Eqs. (3). These differences between the functional and mechanistic usage of the noise may be relevant in particular when dealing with noise-induced transitions (see Sec. III).

B. Feedback: Dichotomous state-dependent noise

The other element of novelty in the dynamics investigated in this paper arises from the feedback between the state x of the system and the random driver. Under the mechanistic interpretation this feedback translates into a dependency of q on x, or of the threshold value s on x (see Fig. 1, dashed lines). We introduce the feedback by assuming that either $p_Q(q)$ or s (or both) depend on the state of the system, namely $p_Q(q) = p_Q(q|x)$ or $s = s(x)$. This implies that also the rates of the DMN depend on x, $k_1(x) = \int_0^s p_Q(q|x)dq$ or $k_1(x) = \int_0^{s(x)} p_Q(q)dq$, and $k_2(x) = 1 - k_1(x)$. Under the functional interpretation, the feedback may produce a state dependency in any of the parameters (k_1, k_2, Δ_1, and Δ_2) of the DMN. However, an eventual x dependency of Δ_1 and/or Δ_2 can be accounted for through a suitable modification of the $g(x)$ function, while the x dependency of k_1 and k_2 intrinsically modifies the dynamical system (the multiplicative noise cannot be factorized). The presence of the state dependency in k_1 and k_2 profoundly affects the x dynamics [Fig. 1(c)], due to the modification of the distribution of the residence times in the states Δ_1 and Δ_2. In fact, this distribution is not exponential as with the standard DMN, consistently with a general property of processes with state-dependent rates (e.g., [18,19]).

The case of a feedback inducing state-dependent thresholds is widespread in environmental systems. For example, dryland vegetation is typically limited by soil moisture; thus random rainfall inputs affect vegetation through the dynamics of soil moisture. Vegetation growth is sustainable only when soil moisture exceeds a certain threshold, otherwise a mortality induced decrease in (live) vegetation biomass occurs. As a result, rainfall inputs determine the switching between stressed and unstressed conditions. The existence of a positive feedback between soil moisture and vegetation makes the switching state dependent: in arid and semiarid environments moister near-surface soils have been consistently found beneath vegetation canopies than in the surrounding bare soil areas [20–22] presumably due to the lower evaporation losses and higher soil infiltration capacity in the subcanopy soils. Through this feedback dryland plants tend to create more favorable conditions for their own survival; thus growth requires less rainwater on well vegetated soils than on soils with only a thin sparse canopy cover. This fact translates into a state dependency in the threshold s of the random driver q (precipitation).

Another example is represented by the dynamics of woody vegetation in semiarid, fire-prone environments. In this case the encroachment of woody plants has been found to be limited by fires (e.g., [11,23]), which, in turn, depend both on ignition and on the presence of grass fuel. In the study of the dynamics of woody vegetation in savannas fire ignition is the random external forcing. Ignition does not act directly on woody plants, in that its effect is mediated by grass fuel availability, which, in turn, is inversely related to woody plant biomass x. Relatively high values of x correspond to a system dominated by woody vegetation, where herbaceous vegetation (i.e., fuel load) is present only in limited amounts. In this system, with the potential for random ignition (e.g., from lightning or land use) being the same, a woodland savanna is less prone to fires than an open savanna with a relatively low tree density. Thus a positive feedback exists between vegetation and the fire pressure, and the dynamics of woody plant biomass can be modeled by Eqs. (1) with state-dependent threshold $s = s(x)$.

To assess the impact of the feedback on the dynamics of the system we will investigate the properties of the pdf of x obtained as solution of the stochastic differential equation (2) with state-dependent parameters. The steady state solution reads [14]

$$p_X(x) = C\left(\frac{1}{f_1(x)} - \frac{1}{f_2(x)}\right)\exp\left[-\int_x \left(\frac{k_1(x')}{f_1(x')} + \frac{k_2(x')}{f_2(x')}\right)dx'\right] \tag{4}$$

with C being a normalization constant calculated by imposing that the integral of $p_X(x)$ in the domain of definition of x

is equal to 1. The zeros of $f_1(x)$ and $f_2(x)$ are the natural boundaries for the dynamics, and represent the limits of the x domain (see also [1]). An alternative representation of the probability density function, which is of interest under the functional interpretation of the DMN, is obtained by using Eq. (3):

$$p_X(x) = \frac{Cg(x)\exp\left\{-\int_x \left(\frac{k_1(x')}{f(x')+\Delta_1 g(x')} + \frac{k_2(x')}{f(x')+\Delta_2 g(x')}\right)dx'\right\}}{[f(x)+\Delta_1 g(x)][f(x)+\Delta_2 g(x)]}. \tag{5}$$

In the following section we will use Eqs. (4) and (5) to investigate the effect of stochastic forcing and feedback on the dynamics expressed by Eqs. (1). Before doing that, we consider the limiting behavior of the steady state pdf of x when the correlation time of noise tends to zero. This applies only in the case when the DMN is used as a simple form of colored noise to perturb the deterministic dynamics $dx/dt = f(x)$ (i.e., functional interpretation); in the reverse case (mechanistic interpretation) the underlying dynamics are intrinsically dichotomic [i.e., the process randomly switches between two alternative dynamics, $dx/dt = f_{1,2}(x)$], and it would be pointless to artificially distort the noise for studying its limiting properties. White shot noise can be obtained as a limit of the dichotomous noise [2] by taking

$$\Delta_1 = \alpha k_1, \quad \Delta_2 = 0, \quad k_1 \to \infty, \quad k_2 = \lambda, \tag{6}$$

where λ and α are the mean frequency and the mean height of the shot noise pulses, respectively. It is interesting to observe that when the same limits (6) are taken in the case of state-dependent dichotomous noise, the x dependency of k_1 is transferred to $\alpha = \alpha(x)$ [e.g., if $k_1(x) = k_1 h(x)$, with $h(x)$ being a generic function of x, then $k_1 \to \infty$ implies $\alpha(x) = \frac{\alpha}{h(x)}$], while the state dependency of $k_2(x)$ translates into a state dependency of $\lambda = \lambda(x) = k_2(x)$.

The steady state distribution of x using the Stratonovich integration rule (which naturally arises when taking the limit from a correlated to a white noise; see [2]) is then obtained as a limit of the dichotomous noise using the (state-dependent) parameter values (6) in Eq. (5),

$$p_{sn}(x) = C\frac{1}{f(x)}\exp\left[-\int_x \frac{f(x') + \lambda(x')\alpha(x')g(x')}{f(x')\alpha(x')g(x')}dx'\right] \tag{7}$$

(see also [24]). Note that $\alpha(x)$ always appears, in Eq. (7), multiplied by $g(x)$: this implies that the state dependency in $k_1(x)$ is simply translated into a modification of the $g(x)$ function by defining $\alpha\bar{g}(x) = \alpha(x)g(x)$. This component of the state dependent noise therefore reduces to a standard multiplicative noise, while a genuine state dependency remains in $\lambda(x)$. In other words, by taking the limits in Eq. (6) from Eq. (2), one obtains the Langevin equation $\frac{dx}{dt} = f(x) + \bar{g}(x)\xi_{sn}$, where ξ_{sn} is a shot noise process with state-dependent rate $\lambda(x)$.

It has also been noted (e.g., [2]) that white Gaussian noise can be obtained as the limit of the dichotomous Markov noise with

$$\Delta_1 = -\Delta_2 = \sqrt{2Dk}, \quad k_1 = k_2 = k \to \infty, \tag{8}$$

where D is the intensity of the noise. In the case of a process $x(t)$ forced by state dependent dichotomous noise, the state-dependency translates into an x-dependent noise intensity, $D = D(x)$. The corresponding steady state pdf is obtained by setting the parameter values (8) in Eq. (5),

$$p_{gn}(x) = C\frac{1}{\sqrt{D(x)}g(x)}\exp\left[\int^x \frac{f(x')}{D(x')g^2(x')}dx'\right]. \tag{9}$$

As in the case of $\alpha(x)$, one has that the square root of $D(x)$ always appears, in Eq. (7), multiplied by $g(x)$, so that the state-dependent noise reduces to a standard multiplicative noise with $\sqrt{D}\bar{g}(x) = \sqrt{D(x)}g(x)$: in fact, the Langevin equation corresponding to Eq. (2) is $\frac{dx}{dt} = f(x) + \bar{g}(x)\xi_{gn}$, where ξ_{sn} is a zero-average Gaussian noise.

III. NOISE-INDUCED TRANSITIONS

In the last few decades the role of noise in stochastic systems has been investigated in relation to its ability to induce new ordered states in dynamical systems. Structural changes in the (stable) states of the system associated with changes in noise intensity are known as noise-induced transitions [3]. A number of studies [3–5] have demonstrated a very counterintuitive effect of noise, i.e., its ability to lead to qualitative changes in the preferential states of a system with respect to those of the underlying deterministic dynamics. The first step towards the identification of noise-induced transitions is therefore the recognition of the correct deterministic counterpart of the dynamics, which can be tricky with the DMN, due to the two different possible interpretations of processes driven by DMN. When the functional interpretation is adopted, the deterministic counterpart is easily found by setting $\xi_{dm}(t) = 0$ in Eq. (2). In this case the deterministic steady states x_{st} are the zeroes of $f(x)$, $f(x_{st}) = 0$.

Under the mechanistic interpretation the picture is somewhat more complicated, and the presence of the feedback further affects the deterministic dynamics (in terms of stationary states). The underlying deterministic dynamics are again obtained by turning to zero the noise variance. If we decrease the variance of the driving force q while maintaining constant its mean q_* in the zero-variance limit, q becomes a constant deterministic value, $q=q_*$. We can distinguish the three cases of (i) no feedback between x and s, (ii) positive feedback (larger x values imply smaller s values), and (iii) negative feedback (larger x values imply larger s values).

(i) When there is no feedback, the deterministic stationary state is determined by the position of q_* relative to s: if $q_* > s$, the (constant) resources are abundant enough to sustain the growth of x expressed by Eq. (1a); the deterministic steady state $x_{st,1}$ is in this case determined by setting $f_1(x_{st,1})=0$. In the reverse case ($q_* < s$) the available resources are scarce, and the dynamics are expressed by Eq. (1b). In this case the deterministic steady state $x_{st,2}$ is found by setting $f_2(x_{st,2})=0$.

(ii) In the case of a positive feedback, the threshold s is a decreasing function of x. In these conditions the deterministic counterpart of the stochastic dynamics depends on the relation between $s(x)$ and q_*. We first define the maximum and minimum possible threshold values by setting in $s(x)$ the relevant domain boundaries, that are $x_{st,2}$ (minimum) and $x_{st,1}$ (maximum), as defined above: the two values $s_1=s(x_{st,1})$ (minimum) and $s_2=s(x_{st,2})$ (maximum) are obtained. If now $q_* < s_1$ the deterministic dynamics monotonically decrease, converging to the stable state $x_{st,2}$. Analogously, if $q_* > s_2$ the system persists in the unstressed state, and the deterministic stable state is $x_{st,1}$. The most interesting situation is the one when $s_2 < q_* < s_1$. In this case the deterministic system is bistable: if the threshold value associated with the initial condition x_0 is smaller than q_* [i.e., $s(x_0) < q_*$], the dynamics of x exhibit a deterministic growth with rate determined by $f_1(x)$. As x grows, $s(x)$ decreases and the system persists in the growth conditions, thereby converging to the steady state $x_{st,1}$. Conversely, if the system is initially in decay (or "stressed") conditions [i.e., $s(x_0) > q_*$], x decreases with rate $f_2(x)$, $s(x)$ increases, and x tends to the steady state $x_{st,2}$.

(iii) In the presence of a negative feedback $s(x)$ increases with x. Also in this case the two limiting threshold values may be defined as $s_1=s(x_{st,1})$ and $s_2=s(x_{st,2})$, with the difference that now s_1 is the maximum and s_2 the minimum value. The deterministic states are then $x_{st,1}$ if $q_* > s_1$, and $x_{st,2}$ if $q_* < s_2$. Again, the most interesting dynamics are found when $s_1 < q_* < s_2$: if the system is initially in the growth state [i.e., $s(x_0) < q_*$], x increases [hence $s(x)$ also increases] until x reaches the value x_*, with $s(x_*)=q_*$. In these conditions the system is stable: in fact, if x exceeds x_*, the state variable, x decreases with rate $f_2(x)$ because $s(x_*) > q_*$. Vice versa, if the system is initially in the stressed (or decay) state, x decreases until it reaches x_* (from above). The stable state of the deterministic system is then x_*.

Once the deterministic counterpart of the dynamics has been identified, it is possible to investigate how noise modifies the stable states of the system. To this end, we analyze the modes and antimodes x_m of the pdf of the process $x(t)$ forced by state-dependent dichotomous noise. These modes can be obtained by setting equal to zero the first-order derivative of Eq. (4) or Eq. (5), depending on the interpretation adopted for the DMN. In the functional interpretation, the modes and antimodes are found from the equation

$$
\begin{aligned}
&f(x_m) + \tau_c \Delta_1 \Delta_2 g(x_m) g'(x_m) + \tau_c(\Delta_1 + \Delta_2) f'(x_m) g(x_m) \\
&\quad + \tau_c \left[2 f(x_m) f'(x_m) - \frac{f^2(x_m) g'(x_m)}{g(x_m)} \right] \\
&\quad + g(x_m) \tau_c [\Delta_1 k_2(x) + \Delta_2 k_1(x)] = 0, \qquad (10)
\end{aligned}
$$

where $\tau_c = \frac{1}{k_1(x)+k_2(x)}$, $g'(x_m) = \frac{dg(x)}{dx}\big|_{x=x_m}$ and $f'(x_m) = \frac{df(x)}{dx}\big|_{x=x_m}$.

The impact of the noise properties on the shape of the pdf clearly appears from Eq. (10). The first four terms in Eq. (10) exist also when the dichotomous noise is state independent [3]. In particular, the first term is independent of the noise parameters and remains even when the noise term in Eq. (2) is turned off. In these conditions the modes and antimodes of $p(x)$ coincide with the stable states of the underlying deterministic dynamics, in that they are given by the condition $f(x_m)=0$; the second term expresses the effect of the multiplicative nature of the noise [i.e., when $g(x) \neq \text{const}$]; the third term results from the asymmetry of the noise (i.e., $\Delta_1 \neq -\Delta_2$), while the fourth term is due to the noise correlation. When the noise is state dependent, the fifth term appears in Eq. (10). This term contributes to the emergence of differences in the stable states (modes) between the stochastic and deterministic dynamics. Notice that when the transition rates are constant (i.e., the noise parameters do not depend on x) the fifth term is zero if the noise is taken, as it is usually done, with a null average value, $\Delta_1 k_2 + \Delta_2 k_1 = 0$.

If the mechanistic interpretation is adopted, it is convenient to write Eq. (10) in terms of the functions $f_1(x)$ and $f_2(x)$,

$$
\begin{aligned}
&\frac{f_1^2(x_m) f_2'(x_m) - f_2^2(x_m) f_1'(x_m)}{f_2(x_m) - f_1(x_m)} - k_1(x_m) f_2(x_m) - k_2(x_m) f_1(x_m) \\
&\quad = 0, \qquad (11)
\end{aligned}
$$

where $f_1'(x_m) = \frac{df_1(x)}{dx}\big|_{x=x_m}$ and $f_2'(x_m) = \frac{df_2(x)}{dx}\big|_{x=x_m}$. It is clear from Eq. (11) that the stable points of the noisy dynamics x_m can be very different from their deterministic counterparts, $x_{st,1}$ and $x_{st,2}$. The role of the state dependency in modifying the stable states is also expressed by the presence of the terms $k_1(x_m)$ and $k_2(x_m)$ in Eq. (11). More precise indications on the occurrence of noise-induced transitions in systems driven by DMN (under the mechanistic interpretation) are provided in the following section.

IV. EXAMPLE

To demonstrate the possible impact of the feedback between noise and the dynamical system, we consider the simple case in which the alternating processes of growth and decay are expressed by two linear functions,

$$
f_1(x) = \alpha(1 - x), \quad f_2(x) = -\alpha x, \qquad (12)
$$

where $\alpha > 0$ determines the rates of growth and decay. The stationary states, $x_{st,1}=1$ and $x_{st,2}=0$, are also the boundaries of the dynamics. We also assume a linear dependence of s on x, $s(x)=s_0+bx$, and a logistic distribution to represent the variability of the resource q, $P_Q(q)=(1+e^{-(q-q_*)/\sigma})^{-1}$, where σ is a scale parameter. The mean and the standard deviation of the distribution are q_* and $\frac{\pi}{\sqrt{3}}\sigma$, respectively. The choice of a linear dependence of s on x and of the use of the logistic distribution are aimed at simplifying the mathematical treatment of the problem, but other choices [i.e., other monotonic forms of the $s(x)$ function, or other probability distributions] would not qualitatively change the results of this section. Under the above assumptions, the transition rates are found as $k_1(x)=1-k_2(x)=(1+e^{-(s_0-q_*+bx)/\sigma})^{-1}$. The corresponding steady-state probability density function from Eq. (4) reads

$$p_X(x) = C_1 x^{(1/\alpha)-1}(1-x)^{-1} \times \exp\left[-\frac{1}{\alpha^2}\int_x \frac{dy}{y(1-y)(1+e^{-(s_0-q_*+by)/\sigma})}\right] \tag{13}$$

with C_1 being the normalization constant calculated by imposing that the integral of $p_X(x)$ in the domain [0,1] is equal to 1.

Figure 2 summarizes the behavior of $p(x)$ across the parameter plane (σ,α) for the case of no feedback [$b=0$, Fig. 2(a)], positive feedback [$b<0$, Fig. 2(b)], or negative feedback [$b>0$, Fig. 2(c)]. The case with $b=0$ [Fig. 2(a)] refers to a situation where $q_*<s_0$, which implies that the deterministic stable state is $x_{st}=0$. For small noise intensities, i.e., for small σ values, the probability distribution is L shaped, i.e., the most probable state is $x=0$. For increasing σ values two different kinds of noise-induced transitions occur: if α is small, i.e., the systems responds slowly to the external forcing, relatively to the rate of switching of the DMN noise, a new mode appears at $x_m=\frac{k_1-\alpha}{1-\alpha}$; if α is relatively large a bifurcation occurs, i.e., the distribution becomes U shaped, with two stable states in $x=0$ and $x=1$. The two curves in Fig. 2(a), marking the separation among the three regimes, are found by considering the behavior of the pdf at the extremes of the domain, and have equations $\alpha=k_1=(1+e^{-(s_0-q_*)/\sigma})^{-1}$ and $\alpha=k_2=1-k_1$.

Figure 2(b) shows the case with a positive feedback ($b<0$). For simplicity, we take $b=2(q_*-s_0)$, which implies that the distribution is symmetrical with respect to $x=0.5$ for any value of the parameters. The distribution is U shaped for low noise intensities, as expected from the bistable behavior of the deterministic counterpart of the dynamics. With increasing values of σ, a first transition occurs, for $\alpha<0.5+\frac{b}{8\sigma}$, with two other modes appearing and the distribution assuming an M shape. If σ is further increased, and $\alpha<k_1(x=1)=k_2(x=0)=(1+e^{(s_0-q_*)/\sigma})^{-1}$, a second transition occurs with a new stationary state in $x_m=0.5$. The stochastic forcing therefore stabilizes the system around a new statistically stable state. This state is clearly noise induced, in that it does not exist in the deterministic counterpart of the process. The ability of noise to turn a bistable deterministic system into a stochastic process with only one stable state (com-

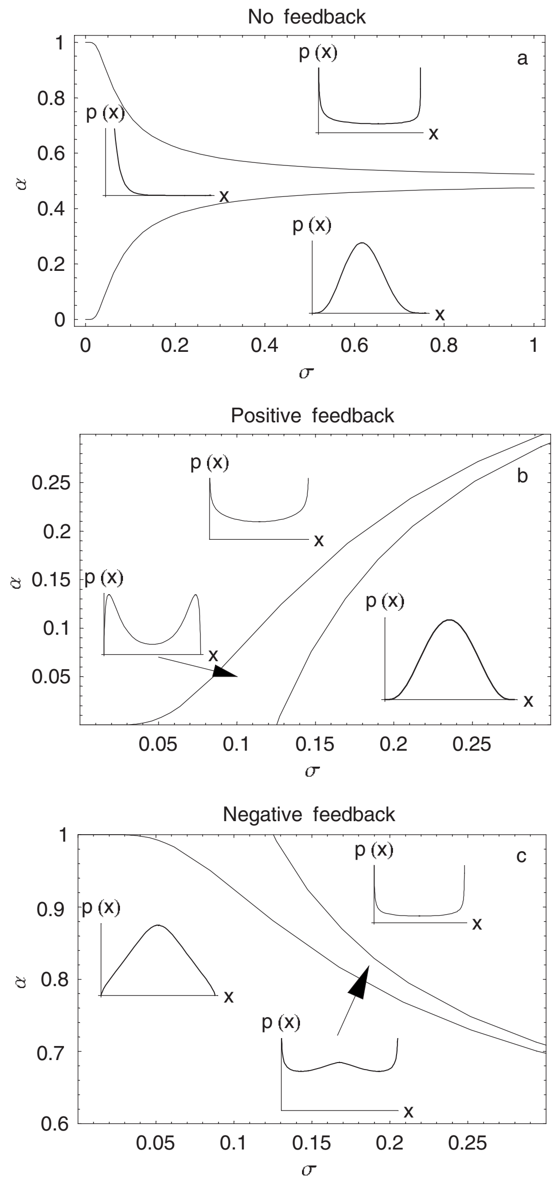

FIG. 2. Shapes of the probability distributions of x as a function of the parameters σ and α. Panel (a) refers to a case with no feedback ($b=0$, $s_0=1.1$, $q_*=1$), panel (b) with positive feedback ($b=-0.5$, $s_0=1.25$, $q_*=1$), and panel (c) with negative feedback ($b=0.5$, $s_0=0.75$, $q_*=1$).

prised between the two stable deterministic states) is known as "noise-induced stability" [16,25,26].

We finally turn to the case with a negative feedback [$b>0$, Fig. 2(c)]. We take again $b=2(q_*-s_0)$ to have a symmetrical distribution. For low values of σ the distribution has a single mode in $x_m=0.5$, which corresponds to the deterministic stable state. For increasing noise intensity, the distribution becomes first W shaped, for $\alpha>k_1(x=1)=k_2(x=0)=(1+e^{(s_0-q_*)/\sigma})^{-1}$, and then U shaped, for $\alpha>0.5+\frac{b}{8\sigma}$. This is a clear example of purely noise-induced bistability, because bistability does not appear in the corresponding deterministic dynamics.

V. CONCLUSIONS

This paper has investigated the effect of (state-dependent) dichotomous Markov noise on a dynamical system. It is well known that noise can induce bistable behavior in systems that do not display any bistable dynamics in the absence of the random driver. Thus noise does not merely induce random fluctuations of the dynamics about its stable states; rather, it creates order by determining the number of stable and unstable states. This qualitative difference between the properties of the stochastic and deterministic dynamics has been usually ascribed to the multiplicative character of noise and to its correlation [1,4]. In this paper we have shown that also state dependency can play a role. This state dependency corresponds to a particular form of multiplicative noise that cannot be factorized, i.e., that cannot be expressed as a byproduct of a function of x and ξ_{dn}. The state dependency may account for possible positive or negative feedback between the random driver and the state of the system. It is found that (i) if the dynamical system has a single deterministic stable point in the absence of the feedback, the deterministic dynamics may become bistable when a positive feedback is introduced. In fact, the positive feedback tends to reduce the rate of switching between the growth and decay states. As a result, the system remains for a longer fraction of time close to the boundaries of the domain, leading to the emergence of U-shaped bimodal distributions. Vice versa, a negative feedback tends to stabilize the system around an intermediate statistically stable state. (ii) Noise may induce bistability in the underlying deterministic dynamics in the presence of a null or negative feedback. (iii) Bistable deterministic dynamics, induced by the positive feedback, may be destroyed by the noise, which tends to stabilize the system around a new intermediate stable state between those of the deterministic dynamics (noise-induced stability). Due to the variety of behaviors they can induce, the interactions between the random forcing and the positive and negative feedback need to be adequately accounted for, since they have been shown to have a strong influence on the system dynamics.

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